

# AN INDUCTIVE JULIA-CARATHÉODORY THEOREM FOR PICK FUNCTIONS IN TWO VARIABLES

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**ABSTRACT.** We study the asymptotic behavior of Pick functions, analytic functions which take the upper half plane to itself. We show that if a two variable Pick function  $f$  has real residues to order  $2N - 1$  at infinity and the imaginary part of the remainder between  $f$  and this expansion is of order  $2N + 1$ , then  $f$  has real residues to order  $2N$  and directional residues to order  $2N + 1$ . Furthermore,  $f$  has real residues to order  $2N + 1$  if and only if the  $2N + 1$ -th derivative is given by a polynomial, thus obtaining a two variable analogue of a higher order Julia-Carathéodory type theorem.

## CONTENTS

|  |    |
|--|----|
| 1. Introduction                            | 1  |
| 1.1. The Löwner class                      | 2  |
| 2. The Agler-McCarthy vector moment theory | 3  |
| 2.1. Some facts about moments              | 5  |
| 3. Proofs of results                       | 7  |
| 3.1. Proof of operator theoretic results   | 7  |
| 3.2. Proof of function theoretic results   | 9  |
| 4. $\mathcal{L}^N \neq \mathcal{L}^{N-}$   | 11 |
| References                                 | 11 |

## 1. INTRODUCTION

The simplest form of classical Julia-Carathéodory theorem, given by Carathéodory[9] and Julia[11], follows.

**Theorem 1.1** (Julia-Carathéodory theorem). *Let  $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$  be an analytic function. The limit  $\lim_{t \rightarrow 1^-} \frac{1 - |f(t)|}{1 - |t|}$  exists if and only if  $\lim_{t \rightarrow 1^-} f(t)$  exists and has modulus 1 and the directional derivative at 1 exists for all directions pointing into the disk.*

The Julia-Carathéodory theorem was extended to higher derivatives by Bolotnikov and Kheifets in [7, 8] and earlier, on the upper half plane, by Nevanlinna in his solution of the Hamburger moment problem[12]. There has been some effort to prove an analogue of the Julia-Carathéodory theorem in several variables in the works of Abate [1, 2], Agler McCarthy, Young [4], Jafari [10], and Włodarczyk [15].

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We are interested in a fusion of the two approaches, that is, an analogue of the Julia-Carathéodory theorem in several variables concerning higher derivatives.

Let  $\Pi$  denote the upper half plane. On  $\Pi$  analogues of the above program exist since  $\Pi$  is conformally equivalent to  $\mathbb{D}$ . We will give an analogue of the Julia-Carathéodory theorem on the domain  $\Pi^2$ . We work in two variables since operator theoretic representation formulas exist for analytic functions  $f : \Pi^2 \rightarrow \overline{\Pi}$ , but do not exist in general due to some classically notorious obstruction [13, 14].

On  $\Pi$ , we have the luxury of the Nevanlinna representation.

**Theorem 1.2** (R. Nevanlinna [12]). *Let  $h : \Pi \rightarrow \mathbb{C}$ . There exists a finite positive Borel measure  $\mu$  on  $\mathbb{R}$  such that*

$$h(z) = \int \frac{1}{t - z} d\mu(t) \quad (1.3)$$

*if and only if  $h$  is analytic, takes values in  $\overline{\Pi}$ , and*

$$\liminf_{s \rightarrow \infty} s |h(is)| < \infty. \quad (1.4)$$

*Moreover, for any Pick function  $h$  satisfying Equation (1.4) the measure  $\mu$  in Equation (1.3) is uniquely determined.*

Notably, the condition (1.4) is conformally related to the limit in the classical Julia-Carathéodory theorem. The Nevanlinna representation can be used to develop a theory of higher order regularity, since essentially questions about regularity are equivalent to elementary questions in real analysis and measure theory. Namely, since

$$\frac{1}{t - z} = - \sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}}$$

questions about regularity at  $\infty$  can be reduced to questions about the existence of moments  $\int t^n d\mu(t)$ . The Nevanlinna representation in several variables is given in terms of operator theory, and so questions there can be reduced to questions some operator theoretic analogue of moments.

**1.1. The Löwner class.** We denote the two variable Pick class, the set of holomorphic functions from  $\Pi^2$  to  $\Pi$ , as  $\mathcal{P}_2$ .

In [3], Agler and McCarthy defined the Löwner class at infinity.

**Definition 1.5.** *The Löwner class at  $\infty$ , denoted  $\mathcal{L}^N$ , is the set of functions  $h \in \mathcal{P}_2$  such that  $\lim_{s \rightarrow \infty} h(is, is) = 0$  and there exists a multi-indexed sequence of real numbers  $(\rho_n)_{|n| \leq 2N-1}$  (here, each  $n = (n_1, n_2)$  for some non-negative integers  $n_1$  and  $n_2$  and  $|n| = n_1 + n_2$ ) such that*

$$h(z) = \sum_{|n| \leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right) \text{ nontangentially.}$$

An asymptotic formula holds *nontangentially* at  $\infty$  if for each  $c \in \mathbb{R}$  the formula holds for all  $z$  large enough satisfying  $\|z\| \leq c \min\{\text{Im}(z_1), \text{Im}(z_2)\}$ .

A weaker notion of regularity is given by the intermediate Löwner class.

**Definition 1.6.** *The intermediate Löwner class at  $\infty$ , denoted  $\mathcal{L}^{N-}$ , is the set of functions  $h \in \mathcal{P}_2$  such that  $\lim_{s \rightarrow \infty} h(is, is) = 0$  and there exists a multi-indexed*

sequence of real numbers  $(\rho_n)_{|n| \leq 2N-2}$  such that

$$h(z) = \sum_{|n| \leq 2N-2} \frac{\rho_n}{z^n} + O\left(\frac{1}{\|z\|^{2N-1}}\right) \text{ nontangentially.}$$

We show that  $\mathcal{L}^N \neq \mathcal{L}^{N-}$  in Section 4.

We examine an inductive relationship between  $\mathcal{L}^{N-1}$ ,  $\mathcal{L}^{N-}$ , and  $\mathcal{L}^N$ , which is given in the following two theorems.

Our first main result describes when a function in  $\mathcal{L}^{N-1}$  is in  $\mathcal{L}^{N-}$ .

**Theorem 1.7.** *Let  $h \in \mathcal{P}_2$ . The following are equivalent:*

- (1)  $h \in \mathcal{L}^{N-}$ .
- (2)  $h \in \mathcal{L}^{N-1}$  and for each  $b \in (\mathbb{R}^+)^2$ ,

$$s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded for large  $s$ .

We prove Theorem 1.7 as Theorem 2.3 in terms of the language of the Agler-McCarthy vector moment theory, which we will discuss later.

Our second main result describes when a function in  $\mathcal{L}^{N-}$  is in  $\mathcal{L}^N$ .

**Theorem 1.8.** *Let  $h \in \mathcal{P}_2$ . The following are equivalent:*

- (1)  $h \in \mathcal{L}^N$ .
- (2)  $h \in \mathcal{L}^{N-}$  and there are residues, not necessarily real,  $\{\rho_n\}_{n \leq 2N-1}$  such that

$$h(z) = \sum_{|n| \leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right)$$

nontangentially.

We prove Theorem 1.8 as Theorem 2.4 in terms of the language of the Agler-McCarthy vector moment theory.

For  $N = 1$  our theorems are conformally equivalent the two-variable Julia-Carathéodory theorem proven on  $\mathbb{D}^2$  in Agler McCarthy, Young [4] where the conformal analogues of  $\mathcal{L}^1$  and  $\mathcal{L}^{1-}$  were called C-points and B-points. An analysis for  $N = 1$  was given on  $\Pi^2$  in Agler, Tully-Doyle, Young[6, 5].

## 2. THE AGLER-McCARTHY VECTOR MOMENT THEORY

A calculus was developed to calculate the residues of functions in  $\mathcal{P}_2$  at  $\infty$  in [6, 3].

**Theorem 2.1** (Type I two variable Nevanlinna representation [6]). *Let  $h \in \mathcal{P}_2$  and suppose that  $sh(is, is)$  is bounded for real  $s$  large enough. Then, there is a separable Hilbert space  $\mathcal{H}$ , an unbounded self-adjoint operator  $A$  on  $\mathcal{H}$ , a positive contraction  $Y$  and a vector  $\alpha \in \mathcal{H}$  such that*

$$\langle (A - z_Y)^{-1} \alpha, \alpha \rangle$$

where  $z_Y = Yz_1 + (1 - Y)z_2$ .

In terms of the above representation, Agler and McCarthy defined *vector moments*, which occur in a way algebraically analogous to the way classical moments occur in the theory of the Nevanlinna representation in one variable.[12].

**Definition 2.2.** *Given a separable Hilbert space  $\mathcal{H}$ , an unbounded self-adjoint operator  $A$  on  $\mathcal{H}$ , a positive contraction  $Y$  and a vector  $\alpha \in \mathcal{H}$ , we say  $A$  has vector moments to order  $N$  denoted  $(R_k)_{k=1}^N$  if*

$$R_k(b) = (b_Y^{-1}A)^{k-1}b_Y^{-1}\alpha$$

*exists for every  $b \in (\mathbb{R}^+)^2$ .*

*If  $R_k$  is a vector-valued polynomial in  $\frac{1}{b_1}$  and  $\frac{1}{b_2}$ , that is, there are vectors  $(\alpha_n)_{|n|=k}$  such that*

$$R_k(b) = \sum_{|n|=k} \frac{1}{b^n} \alpha_n,$$

*we extend  $R_k$  to all of  $\mathbb{C}^2$  via its formula.*

To prove Theorem 1.7 we prove the following equivalence in terms of the Agler-McCarthy vector moment theory.

**Theorem 2.3.** *Let  $h \in \mathcal{P}_2$ . The following are equivalent:*

- (1)  $h \in \mathcal{L}^{N-}$ .
- (2)  $h \in \mathcal{L}^{N-1}$  and for any type I representation of  $h$ ,

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle,$$

*$A$  has real vector  $(Y, \alpha)$ -moments to order  $N - 1$ .*

- (3) For each  $b \in (\mathbb{R}^+)^2$ ,

$$s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

*is bounded for large  $s$ .*

We prove Theorem 2.3 in several parts. The implication (1)  $\Rightarrow$  (2) is given in Proposition 3.1. The implication (2)  $\Rightarrow$  (1) is given in Proposition 3.2. The implication (2)  $\Leftrightarrow$  (3) is given in Proposition 3.4.

Our second result, Theorem 1.8, becomes the following in the language of the Agler-McCarthy moment theory.

**Theorem 2.4.** *Let  $h \in \mathcal{P}_2$ . The following are equivalent:*

- (1)  $h \in \mathcal{L}^N$ .
- (2)  $h \in \mathcal{L}^{N-}$  and for any type I representation of  $h$ ,

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle,$$

*$A$  has vector  $(Y, \alpha)$ -moments to order  $N - 1$  and  $R_{N-1}$  is a vector valued polynomial.*

- (3)  $h \in \mathcal{L}^{N-}$  and there are residues, not necessarily real,  $\{\rho_n\}_{n \leq 2N-1}$  such that

$$h(z) = \sum_{|n| \leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right)$$

*nontangentially.*

Theorem 2.4 is also proven in several parts. (1)  $\Leftrightarrow$  (2) follows directly from the Agler-McCarthy moment theory, specifically their theorem given here as Theorem 2.8, in the light of Theorem 2.3. The implication (1)  $\Leftrightarrow$  (3) is proven as Proposition 3.5.

**2.1. Some facts about moments.** In [3], Agler and McCarthy proved the following:

**Theorem 2.5** (Agler, McCarthy [3]). *Let  $\mathcal{H}$  be a Hilbert space, let  $\alpha \in \mathcal{H}$  and assume that  $A$  and  $Y$  are operators acting on  $\mathcal{H}$ , with  $A$  an unbounded self-adjoint and  $Y$  a positive contraction. The following conditions are equivalent.*

- (i)  *$A$  has finite complex vector  $(Y, \alpha)$ -moments to order  $N - 1$  and for each  $l = 1, \dots, N$  there exist vectors  $\alpha_n$ ,  $|n| = l$  such that*

$$R_l(z) = \sum_{|n|=l} \frac{1}{z^n} \alpha_n$$

*whenever  $z \in \mathbb{C}^2 \setminus \{z \mid z_2 \neq 0, z_1/z_2 \notin (-\infty, 0]\}$ .*

- (ii)  *$A$  has finite real vector  $(Y, \alpha)$ -moments to order  $N - 1$  and for each  $l = 1, \dots, N$  there exist vectors  $\alpha_n$ ,  $|n| = l$  such that*

$$R_l(b) = \sum_{|n|=l} \frac{1}{b^n} \alpha_n \tag{2.6}$$

*whenever  $b \in \mathbb{R}^{+2}$ .*

We also define scalar moments.

**Definition 2.7.** *The  $k$ th real scalar moment is*

$$r_k(b) = \langle R_{\lceil k/2 \rceil}(b), AR_{\lfloor k/2 \rfloor}(b) \rangle.$$

Notably, the  $r_k$  are always real valued when they are defined and furthermore if the  $r_k$  are given by polynomials in  $\frac{1}{b_1}, \frac{1}{b_2}$  then they must have real coefficients.

We will use the following key result about scalar moments.

**Theorem 2.8** (Agler, McCarthy [3]). *A function  $h \in \mathcal{L}^N$  if and only if  $h$  has a type I representation*

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle,$$

*such that  $A$  has polynomial vector  $(Y, \alpha)$ -moments to order  $N - 1$ . Moreover,*

$$r_k(z) = - \sum_{|n|=k} \frac{\rho_n}{z^n}$$

*where  $\rho_n$  are as in Definition 1.5.*

The following telescoping lemma gives a formula that will let us prove the main results.

**Lemma 2.9.** *Let  $h \in \mathcal{P}_2$  with type I representation*

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle,$$

be such that  $A$  has vector  $(Y, \alpha)$ -moments to order  $N - 1$  and scalar moments up to order  $2N - 1$ . Let  $b \in (\mathbb{R}^+)^2$ . Let  $X_b = b_Y^{-1/2} A b_Y^{-1/2}$ . Let  $\beta_k = X_b^k b_Y^{-1/2} \alpha$ . Then,

$$h(isb) + \sum_{k=1}^{2N-1} r_k(isb) = \frac{\langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}}.$$

*Proof.* Note  $b_Y^{-1/2}$  exists since  $b_Y$  is strictly positive. Note that the expressions for  $r_{2k-1}$  and  $r_{2k}$  in the notation of the lemma become:

$$\begin{aligned} r_{2k-1}(isb) &= (is)^{-(2k-1)} \langle \beta_k, \beta_k \rangle \\ r_{2k}(isb) &= (is)^{-2k} \langle \beta_{k-1}, \beta_k \rangle. \end{aligned}$$

The proof will go by induction. When  $N = 1$ ,

$$\begin{aligned} h(isb) + r_1(isb) &= \langle (A - isb_Y)^{-1} \alpha, \alpha \rangle + \langle (isb_Y)^{-1} \alpha, \alpha \rangle \\ &= \langle (X_b - is)^{-1} \beta_0, \beta_0 \rangle + \langle (is)^{-1} \beta_0, \beta_0 \rangle \\ &= \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_0, \beta_0 \rangle. \end{aligned}$$

So we are done. Now suppose, by induction,

$$h(isb) + \sum_{k=1}^{2N-1} r_{2k-1}(isb) = \frac{\langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}}$$

and additionally we have vector  $(Y, \alpha)$ -moments to order  $N$  and scalar  $(Y, \alpha)$ -moments to order  $2N + 1$ . So,

$$\begin{aligned} h(isb) + \sum_{k=1}^{2N+1} r_k(isb) &= \frac{\langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_{N-1}, \beta_N \rangle}{(is)^{2N}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-1} + (is)^{-2} X_b] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-2} (X_b + is)] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-2} (X_b + is)(X_b - is)(X_b - is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + ((is)^{-2} X_b^2 - 1)(X_b - is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2(N-1)}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [X_b^2 (X_b - is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle}{(is)^{2N}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1}] X_b \beta_{N-1}, X_b \beta_{N-1} \rangle}{(is)^{2N}} + \frac{\langle \beta_N, \beta_N \rangle}{(is)^{2N+1}} \\ &= \frac{\langle [(X_b - is)^{-1} + (is)^{-1}] \beta_N, X_b \beta_N \rangle}{(is)^{2N}}. \end{aligned}$$

This concludes the proof.  $\square$

## 3. PROOFS OF RESULTS

**3.1. Proof of operator theoretic results.** First we endeavor to prove the equivalence of (1) and (2) in Theorem 2.3. We separate the proof into two parts.

**Proposition 3.1.** *Let  $h \in \mathcal{P}_2$  with type I representation*

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle.$$

*If  $h \in \mathcal{L}^{N-}$ , then  $h \in \mathcal{L}^{N-1}$  and  $A$  has vector  $(Y, \alpha)$ -moments to order  $N - 1$ .*

*Proof.* Suppose  $h \in \mathcal{L}^{N-}$ . Then,  $h \in \mathcal{L}^{N-1}$ . We will show  $A$  has real vector  $(Y, \alpha)$ -moments to order  $N - 1$ . This is sufficient by Theorem 2.5. By Theorem 2.8,

$$h(isb) + \sum_{k=1}^{2N-3} r_k(isb) = h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n},$$

and  $A$  has  $(Y, \alpha)$ -moments to order  $N - 2$ . So by Lemma 2.9, adopting its notation,

$$(is)^{2N-1} \left[ h(isb) - \sum_{|n| \leq 2N-2} \frac{\rho_n}{(isb)^n} \right] = (is)^3 \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle - is \sum_{|n|=2N-2} \frac{\rho_n}{b^n}.$$

Since  $h \in \mathcal{L}^{N-}$ , for some  $C > 0$ ,

$$\left| (is)^{2N-1} \left[ h(isb) - \sum_{|n| \leq 2N-2} \frac{\rho_n}{(isb)^n} \right] \right| \leq C$$

So,

$$\left| (is)^3 \langle [(X_b - is)^{-1} - (is)^{-1}] \beta_{N-2}, \beta_{N-2} \rangle - (is) \sum_{|n|=2N-2} \frac{\rho_n}{b^n} \right| \leq C.$$

Taking the real part preserves this inequality. Thus,

$$\left| \operatorname{Re} (is)^3 \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-2}, \beta_{N-2} \rangle \right| \leq C.$$

Simplifying,

$$\begin{aligned} \left| \operatorname{Re} (is)^2 \left\langle \frac{X_b}{X_b - is} \beta_{N-2}, \beta_{N-2} \right\rangle \right| &\leq C \\ \left| \operatorname{Re} s^2 \left\langle \frac{X_b^2 + isX_b}{X_b^2 + s^2} \beta_{N-2}, \beta_{N-2} \right\rangle \right| &\leq C \\ \left\langle \frac{s^2 X_b^2}{X_b^2 + s^2} \beta_{N-2}, \beta_{N-2} \right\rangle &\leq C \end{aligned}$$

By the spectral theorem, there is a measure  $\mu$  so that,

$$\left\langle \frac{s^2 X_b^2}{X_b^2 + s^2} \beta_{N-2}, \beta_{N-2} \right\rangle = \int \frac{s^2 x^2}{x^2 + s^2} |\beta_{N-2}(x)|^2 d\mu(x).$$

Note the integrand is monotone increasing in  $s$ , so apply monotone convergence theorem to get

$$\int |x \beta_{N-2}(x)|^2 d\mu(x) = \int |\beta_{N-1}(x)|^2 d\mu(x)$$

exists and is finite. So  $X_b \beta_{N-2} \in \operatorname{Dom} X_b$ . That is,  $(b_Y^{-1/2} A b_Y^{-1/2})^{N-1} b_Y^{-1/2} \alpha \in \operatorname{Dom} b_Y^{-1/2} A b_Y^{-1/2}$ . So,  $(A b_Y^{-1})^{N-1} \in \operatorname{Dom} A$ . Thus,  $A$  has  $(Y, \alpha)$ -moments to order  $N - 1$ .  $\square$

The other direction goes as follows.

**Proposition 3.2.** *Let  $h \in \mathcal{P}_2$  with type I representation*

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle.$$

*Then, if  $h \in \mathcal{L}^{N-1}$  and  $A$  has vector  $(Y, \alpha)$ -moments to order  $N-1$ , then  $h \in \mathcal{L}^{N-}$ .*

*Proof.* Suppose  $h \in \mathcal{L}^{N-1}$  and  $h$  has  $(Y, \alpha)$ -moments to order  $N-1$ . By Theorem 2.5,

$$z_Y^{-1}(Az_Y^{-1})^{N-2}\alpha = R_{N-1}(z) = \sum_{|n|=N-1} \frac{1}{z^n} \alpha_n.$$

Since we have  $(Y, \alpha)$ -moments to order  $N-1$ ,

$$(Az_Y^{-1})^{N-1}\alpha = AR_{N-1}(z) = A \sum_{|n|=N-1} \frac{1}{z^n} \alpha_n$$

is well defined. Note, by linear independence of monomials, each  $\alpha_n \in \text{Dom} A$ . Thus,

$$(Az_Y^{-1})^{N-1}\alpha = AR_{N-1}(z) = \sum_{|n|=N-1} \frac{1}{z^n} A\alpha_n.$$

So,

$$\begin{aligned} r_{2N-2}(z) &= \langle AR_{N-1}(z), R_{N-1}(\bar{z}) \rangle \\ &= \sum_{|m|=N-1} \sum_{|n|=N-1} \frac{1}{z^n \bar{z}^m} \langle A\alpha_n, \alpha_m \rangle = \sum_{|n|=2N-2} \frac{\rho_n}{z^n}, \end{aligned}$$

where  $\rho_n = \sum_{n+m=2N-2} \langle A\alpha_n, \alpha_m \rangle$ . Note that if  $b \in (\mathbb{R}^+)^2$ ,

$$r_{2N-2}(isb) = \frac{1}{(is)^{2N-2}} = \sum_{|n|=2N-2} \frac{\rho_n}{z^n}$$

is real valued. Thus, by linear independence of monomials, each  $\rho_n$  is real valued. So

$$\begin{aligned} h(z) - \sum_{l=1}^{2N-2} r_l(z) &= \langle (A - z_Y)^{-1} AR_{N-1}(z), R_{N-1}(\bar{z}) \rangle \\ \|z\|^{2N-1} (h(z) - \sum_{l=1}^{2N-2} r_l(z)) &= \|z\|^{2N-1} \langle (A - z_Y)^{-1} AR_{N-1}(z), R_{N-1}(\bar{z}) \rangle. \end{aligned}$$

Now notice

$$\|z\|^{2N-1} |h(z) - \sum_{l=1}^{2N-2} r_l(z)| \leq \|z\| \| (A - z_Y)^{-1} \| \|z\|^{N-1} \|AR_{N-1}(z)\| \|z\|^{N-1} \|R_{N-1}(z)\|$$

is nontangentially bounded. So,  $h \in \mathcal{L}^{N-}$ .  $\square$

This concludes the proof of the equivalence of (1) and (2) in Theorem 2.3.



**3.2. Proof of function theoretic results.** We now seek to prove the implication (1)  $\Leftrightarrow$  (3) in Theorem 2.3 and Theorem 2.4.

We begin with the following lemma which will allow us to prove (1)  $\Leftrightarrow$  (3) for Theorem 2.3.

**Lemma 3.3.** *Let  $h \in \mathcal{P}_2$ . Suppose  $h \in \mathcal{L}^{N-1}$  and for each  $b \in (\mathbb{R}^+)^2$ , for large  $s$ ,*

$$(is)^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

*is bounded. Then*

$$r_{2N-1}(b) = \lim_{s \rightarrow \infty} (is)^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right].$$

*Proof.* Suppose  $h \in \mathcal{L}^{N-1}$  and for each  $b \in (\mathbb{R}^+)^2$ ,

$$J_b(s) := s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded. Let  $h$  have a type I representation  $h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle$ . We will show  $A$  has vector  $(Y, \alpha)$ -moments to order  $N-1$  and apply the equivalence of 1 and 2 in Theorem 2.3. By Lemma 2.9, adopting its notation,

$$h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} = (is)^{-2(N-2)} \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-2}, \beta_{N-2} \rangle.$$

With this substitution,

$$J_b(s) = (is)^{2N-1} \operatorname{Im} (is)^{-2(N-2)} \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-2}, \beta_{N-2} \rangle.$$

Simplified, we obtain

$$J_b(s) = \langle \frac{-s^2 X_b^2}{X_b^2 + s^2} \beta_{N-2}, \beta_{N-2} \rangle.$$

Applying the spectral theorem and monotone convergence theorem as in the proof of the equivalence on (1) and (2) in Theorem 2.3, we get

$$\lim_{s \rightarrow \infty} (is)^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right] = -\|\beta_{N-1}\|^2 = r_{2N-1}(b).$$

□

We now prove (1)  $\Leftrightarrow$  (3) for Theorem 2.3.

**Proposition 3.4.** *Let  $h \in \mathcal{P}_2$ . Then,  $h \in \mathcal{L}^{N-}$  if and only if  $h \in \mathcal{L}^{N-1}$  and for each  $b \in (\mathbb{R}^+)^2$ , for large  $s$ ,*

$$s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

*is bounded.*

*Proof.* Suppose  $h \in \mathcal{L}^{N-}$ . The term  $\sum_{|n|=2N-2} \frac{\rho_n}{(isb)^n}$  is real. So,

$$s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right] = s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-2} \frac{\rho_n}{(isb)^n} \right],$$

which is bounded since  $h \in \mathcal{L}^{N-}$ .

On the other hand, suppose  $h \in \mathcal{L}^{N-1}$  and for each  $b \in (\mathbb{R}^+)^2$ ,

$$s^{2N-1} \operatorname{Im} \left[ h(isb) - \sum_{|n| \leq 2N-3} \frac{\rho_n}{(isb)^n} \right]$$

is bounded. By Theorem 3.3, we have scalar moments to order  $2N-1$  and thus vector  $(Y, \alpha)$ -moments. So by the equivalence of (1) and (2) in Theorem 2.3, we are done.  $\square$

The following finishes the proof of Theorem 2.4 by showing that We now prove (1)  $\Leftrightarrow$  (3).

**Proposition 3.5.** *Let  $h \in \mathcal{P}_2$ . Then,  $h \in \mathcal{L}^N$  if and only if  $h \in \mathcal{L}^{N-}$  and there are residues, not necessarily real,  $\{\rho_n\}_{n \leq 2N-1}$  such that*

$$h(z) = \sum_{|n| \leq 2N-1} \frac{\rho_n}{z^n} + o\left(\frac{1}{\|z\|^{2N-1}}\right)$$

*nontangentially.*

*Proof.* The forward direction is true by definition.

On the other hand, suppose  $h \in \mathcal{L}^{N-}$  and the residues exist. Let  $b \in (\mathbb{R}^+)^2$ . Let  $h$  have a type I representation

$$h(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle.$$

By Lemma 2.9, adopting its notation,

$$h(isb) + \sum_{k=1}^{2N-1} r_k(isb) = (is)^{-2(N-1)} \langle [(X_b - is)^{-1} + (is)^{-1}] \beta_{N-1}, \beta_{N-1} \rangle.$$

Multiply through by  $(is)^{2N-1}$ .

$$(is)^{2N-1} \left[ h(isb) + \sum_{k=1}^{2N-1} r_k(isb) \right] = \langle [is(X_b - is)^{-1} + 1] \beta_{N-1}, \beta_{N-1} \rangle.$$

So,

$$(is)^{2N-1} \left[ h(isb) + \sum_{k=1}^{2N-1} r_k(isb) \right] = \langle [\frac{-s^2}{X_b^2 + s^2} + 1] \beta_{N-1}, \beta_{N-1} \rangle.$$

Applying the spectral theorem and taking limits gives

$$\lim_{s \rightarrow \infty} (is)^{2N-1} \left[ h(isb) + \sum_{k=1}^{2N-1} r_k(isb) \right] = -\|\beta_{N-1}\|^2 + \|\beta_{N-1}\|^2 = 0.$$

Now

$$\lim_{s \rightarrow \infty} (is)^{2N-1} \left[ h(isb) + \sum_{k=1}^{2N-1} r_k(isb) - h(isb) + \sum_{|n| \leq 2N-1} \frac{\rho_n}{(isb)^n} \right] = 0.$$

Applying Theorem 2.8,

$$\lim_{s \rightarrow \infty} (is)^{2N-1} \left[ r_{2N-1}(isb) + \sum_{|n|=2N-1} \frac{\rho_n}{(isb)^n} \right] = 0.$$

Simplifying,

$$\lim_{s \rightarrow \infty} r_{2N-1}(b) + \sum_{|n|=2N-1} \frac{\rho_n}{b^n} = 0,$$

that is,

$$r_{2N-1}(b) = - \sum_{|n|=2N-1} \frac{\rho_n}{b^n}.$$

□

#### 4. $\mathcal{L}^N \neq \mathcal{L}^{N-}$

Now we give an example that shows the hierarchy of Löwner classes in two variables at infinity does not collapse, that is,  $\mathcal{L}^N \neq \mathcal{L}^{N-}$ , which was shown for the case  $N = 1$  in [4]. That  $\mathcal{L}^N \neq \mathcal{L}^{N-}$  is in stark contrast to the theory in one variable where the classes are identical[11, 9].

Let  $\mathcal{H} = l^2(Z_{2(n-1)})$ , and  $\pi : Z_{2(n-1)} \rightarrow B(l^2(Z_{2(n-1)}))$  a left regular representation i.e.  $\pi(j)e_i = e_{j+i}$ . Let  $A = [\pi(1) + \pi(-1)]$  and  $Y$  be a diagonal matrix satisfying  $Ye_i = e_i$ , for  $i \neq n$ , and  $Ye_{n-1} = te_{n-1}$ . Let  $\alpha = e_0$ . Let  $f$  be the Pick function defined by

$$f(z) = \langle (A - z_Y)^{-1} \alpha, \alpha \rangle.$$

Recall  $R_k(z) = (z_Y)^{-1} (Az_Y^{-1})^{k-1} e_0$ . If  $k < n$ , it can be shown inductively that

$$R_k(z) = z_1^{-k} \sum_{l=0}^{k-1} \binom{k-1}{l} e_{-(k-1)+2l}.$$

Furthermore,

$$AR_{n-1}(z) = z_1^{-(n-1)} \sum_{l=0}^{n-1} \binom{n-1}{l} e_{-(n-1)+2l}$$

and

$$R_n(z) = \frac{1}{tz_1 + (1-t)z_2} z_1^{-(n-1)} e_{n-1} + z_1^{-n} \sum_{l=0}^{n-2} \binom{n-1}{l} e_{-(n-1)+2l}.$$

So,  $r_{2n-1}(z) = \langle R_n(z), AR_{n-1}(\bar{z}) \rangle$  is not a polynomial, but for  $k < 2n-1$ ,  $r_k$  is a polynomial. That is,  $f \in \mathcal{L}^{N-}$ , but  $f \notin \mathcal{L}^N$ .

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